A SHARP ESTIMATE OF WEIGHTED DYADIC SHIFTS OF COMPLEXITY 0 AND 1

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Abstract. A simple shortcut to proving sharp weighted estimates for the Martingale Transform and for the Hilbert transform is presented. It is a unified proof for these both transforms.

1. Introduction

Let $\sigma := w^{-1}$.

Notations. We call a shift by n generations, or SH_n any sub-bilinear operator of the following form

$$(SH_n\, f_1, f_2) = \sum_{J\subset I, |J|=2^{-n}|I|} 2^{-\frac{n}{2}} c_{IJ} |(f_1, h_I)||(f_2, h_J)|\,,$$

where $|c_{IJ}| \le 1$.

Theorem 1.1.

$$(SH_1\, f_1, f_2) \le C\, [w]_{A_2} \|f_1\|_w \|f_2\|_\sigma\,.$$

Proof. Let

$$Q := [w]_{A_2}\,.$$

We know from [26], [40] that

Theorem 1.2. *There exists a function B_Q of 6 variables (X,Y,x,y,u,v) defined in $\Omega_Q := \{(X,Y,x,y,u,v) > 0 : x^2 \le Xv, y^2 \le Yu, 1 \le uv \le Q\}$ such that*

$$B_Q(X,Y,x,y,u,v) \le C\, Q\, (X+Y)\,, \tag{1.1}$$

and

$$d^2 B_Q(X,Y,x,y,u,v) \ge |dx||dy|\,. \tag{1.2}$$

In particular, one can conclude that having two points $a = (a_1, ..., a_6)$, $b = (b_1, ..., b_6)$ in Ω_Q connected by segment $[a, b]$ **lying entirely inside** Ω_Q one can introduce the parametrization $c(t) = at + b(1-t)$, consider

$$q(t) = B_Q(c(t))$$

and claim, using (1.2) that

$$-\, q''(t) \ge |a_3 - b_3||a_4 - b_4|\,. \tag{1.3}$$

We will need the same thing for some other segments $[a, b]$ **not lying entirely inside** Ω_Q (but with $a, b \in \Omega_Q$).

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The problem is of course that Ω_Q is not convex.

Now let us apply B_{40Q}. We choose I and put

$$b := (\langle f_1^2 w\rangle_I, \langle f_2^2\sigma\rangle_I, \langle f_1\rangle_I, \langle f_2\rangle_I, \langle w\rangle_I, \langle\sigma\rangle_I)\,,$$

$$b_+ = (\langle f_1^2 w\rangle_{I_+}, \langle f_2^2\sigma\rangle_{I_+}, \langle f_1\rangle_{I_+}, \langle f_2\rangle_{I_+}, \langle w\rangle_{I_+}, \langle\sigma\rangle_{I_+})\,,$$

$$b_- = (\langle f_1^2 w\rangle_{I_-}, \langle f_2^2\sigma\rangle_{I_-}, \langle f_1\rangle_{I_-}, \langle f_2\rangle_{I_-}, \langle w\rangle_{I_-}, \langle\sigma\rangle_{I_-})\,,$$

$$b_{ij} = (\langle f_1^2 w\rangle_{I_{ij}}, \langle f_2^2\sigma\rangle_{I_{ij}}, \langle f_1\rangle_{I_{ij}}, \langle f_2\rangle_{I_{ij}}, \langle w\rangle_{I_{ij}}, \langle\sigma\rangle_{I_{ij}})\,,$$

where $i, j = \pm$.

We want to estimate from below

$$D := B_{40Q}(b) - \frac{1}{4}(\sum_{i,j=\pm} B_{40Q}(b_{ij}) = A + B + C\,,$$

where

$$A := B_{40Q}(b) - \frac{1}{2}(B_{40Q}(b_+) + B_{40Q}(b_-))\,,$$

$$B := \frac{1}{2}(B_{40Q}(b_+) - \frac{1}{2}(B_{40Q}(b_{++}) + B_{40Q}(b_{+-})))\,,$$

$$C := \frac{1}{2}(B_{40Q}(b_-) - \frac{1}{2}(B_{40Q}(b_{-+}) + B_{40Q}(b_{--})))\,.$$

Let $b = (\cdot, \cdot, x, y, \cdot, \cdot)$, $b_+ = (\cdot, \cdot, x+\alpha, y+\lambda, \cdot, \cdot)$, $b_- = (\cdot, \cdot, x-\alpha, y-\lambda, \cdot, \cdot)$,

$$b_{++} = (\cdot, \cdot, x+\alpha+\beta_1, y+\lambda+\delta_1, \cdot, \cdot)\,, b_{+-} = (\cdot, \cdot, x+\alpha-\beta_1, y+\lambda-\delta_1, \cdot, \cdot)\,,$$

$$b_{-+} = (\cdot, \cdot, x-\alpha+\beta_2, y-\lambda+\delta_2, \cdot, \cdot)\,, b_{--} = (\cdot, \cdot, x-\alpha-\beta_2, y-\lambda-\delta_2, \cdot, \cdot)\,.$$

We do not know the signs of $\alpha, \lambda, \beta_1, \beta_2, \delta_1, \delta_2$.

We want to show that there exists an absolute positive constant c such that

$$D \geq c\,|\alpha|(|\delta_1| + |\delta_2|)\,. \tag{1.4}$$

Consider several cases. First of all notice that not only all b, b_-, b_+, b_{ij} are in Ω_Q but the segments $[b, b_{ij}]$ are in Ω_{40Q}. This follows from the following geometric lemma.

Lemma 1.3. *Let three point A, B, C be in Ω_Q and let $M = \frac{A+B}{2}$. Assume $[A, B] \subset \Omega_Q$ and $[C, M] \subset \Omega_Q$. Then $[C, A], [C, B] \subset \Omega_{40Q}$.*

Proof. Let's prove the statement for $[C, A]$.

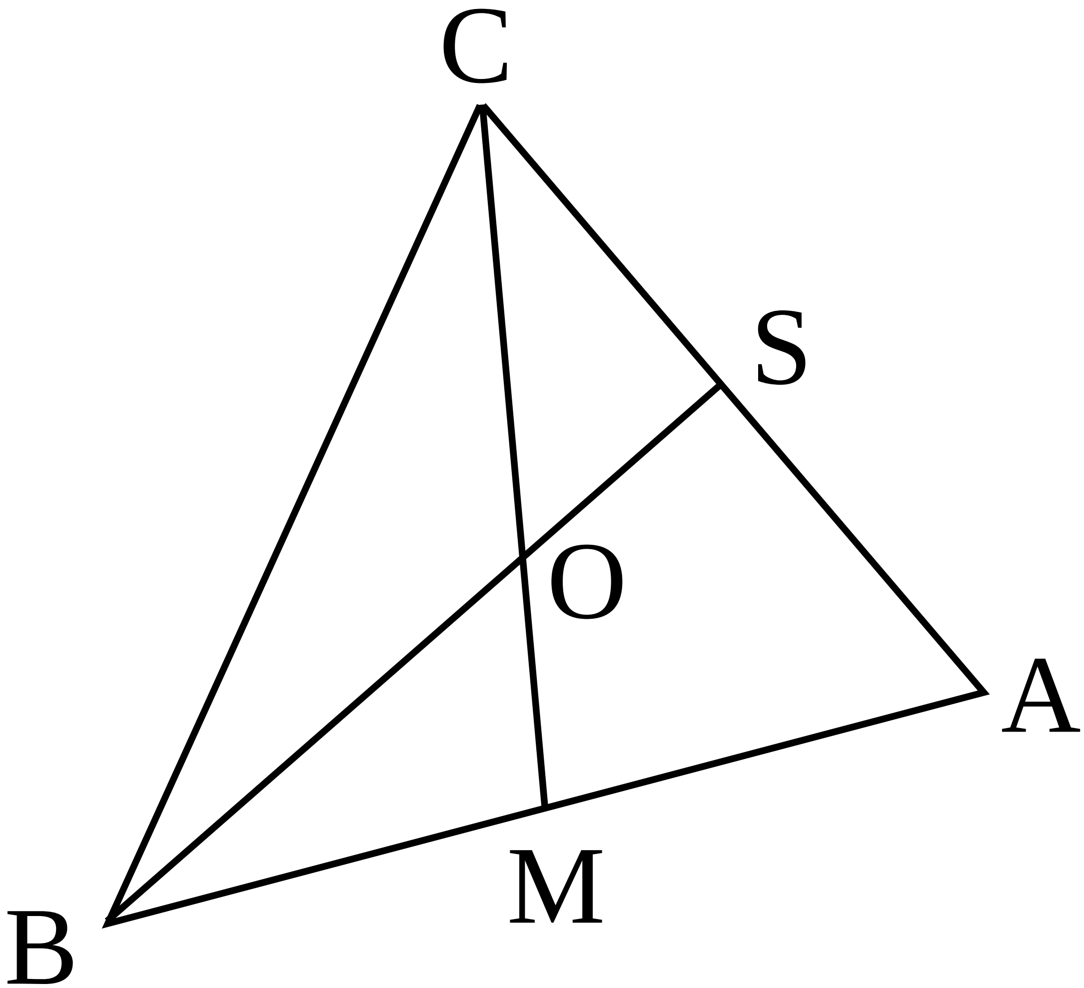
Case 1: $C_1 \leqslant A_1$, $C_2 \leqslant A_2$. Then there is nothing to prove, since if we have a line segment with positive slope, who's endpoints are in Ω_Q, then the whole segment lies in Ω_Q.

Case 2: $C_1 \geqslant A_1$ or $C_2 \geqslant A_2$. Without loss of generality, assume $C_1 \geqslant A_1$.

Denote $S = \frac{A+C}{2}$ — the middle of $[A, C]$. Denote also $O = [C, M] \cap [B, S]$. Since $[C, M]$ and $[B, S]$ are two medians of the triangle ABC, we have that O is the center of ABC. Therefore,

$$O = \frac{1}{3}B + \frac{2}{3}S, \tag{1.5}$$

$$O = \frac{1}{3}C + \frac{2}{3}M. \tag{1.6}$$



Therefore, for $k \in \{1,2\}$ we have

$$O_k \geqslant \frac{2}{3} M_k, \tag{1.7}$$

$$O_k \geqslant \frac{1}{3} C_k. \tag{1.8}$$

On the other hand,

$$S_1 = \frac{A_1 + C_1}{2} \leqslant C_1 \leqslant 3O_1.$$

Therefore,

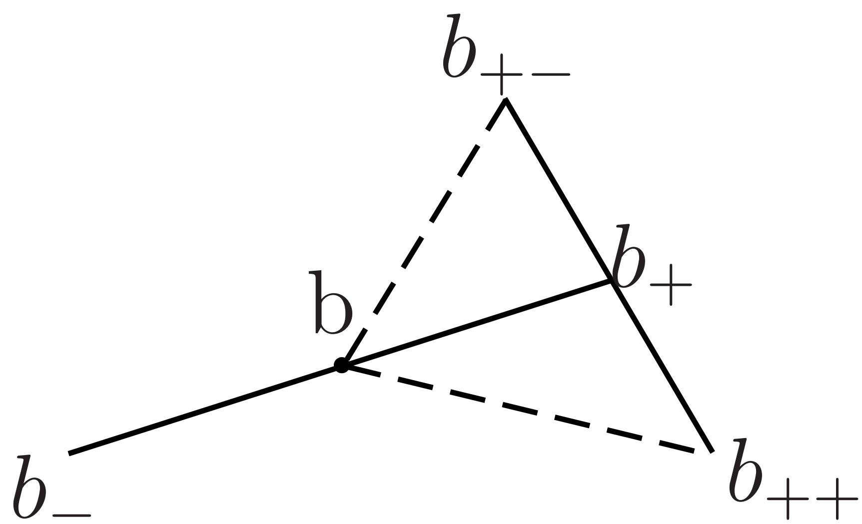
$$S_1 S_2 \leqslant 3O_1 \cdot \frac{3}{2} O_2 = \frac{9}{2} O_1 O_2.$$

But $O \in [C, M] \subset \Omega_Q$, so

$$S_1 S_2 \leqslant \frac{9}{2} Q.$$

Therefore, $S \in \Omega_{\frac{9}{2}Q}$, and so are A and C. Thus, $[A, C] \in \Omega_{40Q}$, which finishes the proof. $\square$

The statement for segments $[b, b_{ij}]$ follows from this lemma. Indeed, we have a triangle $bb_{++}b_{+-}$ such that $[b_{++}, b_{+-}] \subset \Omega_Q$ and, moreover, since endpoints and the middle of the line segment b_-bb_+ are in Ω_Q, we conclude that $[b, b_+] \in \Omega_{2Q}$. Therefore, the median of mentioned triangle is in Ω_{2Q}, thus, all sides are in Ω_{40Q}.



Lemma 1.4. *Let points P, $P_i, i = 1, 2, 3, 4$ be in Ω_Q and P be a baricenter of P_i. Then all segments $[P, P_i]$ are in Ω_{40Q}.*

Now fix i, j, say, $i = +, j = -$. Consider function

$$f_{+-}(t) = B_{40Q}(tb_{+-} + (1 - t)b)$$

and write

$$f_{+-}(0) - f_{+-}(1) = -f'(0) - \frac{1}{2}f''(\xi) = -\nabla B_{40Q}(b) \cdot (b_{+-} - b) + \frac{1}{2}|x + \alpha - \beta_1 - x||y + \lambda - \delta_1 - y|\,.$$

This is because of Theorem 1.2.

We do this for all f_{ij}, $i = \pm, j = \pm$, add and divide by 4. Then we get the first estimate on D:

$$D \ge -\nabla B_{40Q}(b) \cdot (\frac{1}{4}(b_{+-} + b_{++} + b_{--} + b_{-+}) - b)$$

$$+ \frac{1}{2}((|\alpha - \beta_1||\lambda - \delta_1| + |\alpha + \beta_1||\lambda + \delta_1|) + (|\alpha - \beta_2||\lambda - \delta_2| + |\alpha + \beta_2||\lambda + \delta_2|))\,. \tag{1.9}$$

The first term is zero. If we have the case that $|\beta_1| \le \frac{1}{2}|\alpha|$ and $|\beta_2| \le \frac{1}{2}|\alpha|$, then we get from the first bracket of the second term at least $|\alpha||\delta_1|$, and from the second bracket at least $|\alpha||\delta_1|$. In this case (1.4) is proved.

Suppose now that $|\beta_1| \ge \frac{1}{2}|\alpha|$ and $|\beta_2| \ge \frac{1}{2}|\alpha|$. Then we notice that $D = A + B + C$. Moreover, $A \ge 0$ as B_{40Q} is concave, and $[b_-, b_+] \subset \Omega_{40Q}$ (see Lemma 1.4), point b being the center of this segment. On the other hand, by Theorem 1.2

$$2B \ge B_{40Q}(b_+) - \frac{1}{2}(B_{40Q}(b_{++}) + B_{40Q}(b_{+-})) \ge c\,|\beta_1||\delta_1| \ge \frac{c}{2}|\alpha||\delta_1|$$

by our assumption. Symmetrically we will have

$$2c \geq B_{40Q}(b_-) - \frac{1}{2}(B_{40Q}(b_{-+}) + B_{40Q}(b_{--})) \geq c\,|\beta_2||\delta_2| \geq \frac{c}{2}|\alpha||\delta_2|\,.$$

Combining the last two inequalities we also have

$$D = A + B + C \geq c'\,|\alpha|(|\delta_1| + |\delta_2|)\,,$$

which is (1.4) we want.

Now suppose $|\beta_1| \leq \frac{1}{2}|\alpha|$ and $|\beta_2| \geq \frac{1}{2}|\alpha|$. Then we write $2D = D+A+B+C$. We estimate D by (1.9), omitting the the second (positive) bracket of the second term, and writing for the first bracket of the second term the following estimate:

$$D \geq |\alpha||\delta_1|\,.$$

We again use that $A \geq 0$ and also use that $B \geq 0$ by the same concavity and the fact that $[b_{+-}, b_{++}] \subset \Omega_{40Q}$.

On the other hand, by Theorem 1.2

$$2C \geq B_{40Q}(b_-) - \frac{1}{2}(B_{40Q}(b_{-+}) + B_{40Q}(b_{--})) \geq c\,|\beta_2||\delta_2| \geq \frac{c}{2}|\alpha||\delta_2|$$

by our assumption $|\beta_2| \geq \frac{1}{2}|\alpha|$. Now combining $2D = D + A + B + C$ and the last two inequalities we get (1.4).

We are left with the fourth case: $|\beta_1| \geq \frac{1}{2}|\alpha|$ and $|\beta_2| \leq \frac{1}{2}|\alpha|$. But it is totally symmetric to the previous case. So (1.4) is always proved.

Now we repeat the usual Bellman function summation over dyadic tree (we have above the inequality for the node I, we repeat it for nodes I_+, I_- et cetera). In other words we use integration of discrete Laplacian and discrete Green's formula to get (we use also (1.1) of course):

$$\frac{1}{|I|}\sum_{J\subset I} |L||\Delta_J f_1|(|\Delta_{J_-} f_2| + \Delta_{J_+} f_2|) \leq C\,40\,Q\,(\langle f^2 w\rangle_I + \langle g^2\sigma\rangle_I)\,. \tag{1.10}$$

Our Theorem 1.1is completely proved.

□

2. Points over i's

We gave a simple proof of linear estimate of any shift of complexity 1. So, for example, it gives the way to deduce Stefanie's result from [26]. Below we give a very simple proof of [40]. This is up to the existence of B_Q. In the next section we give a proof of such an existence.

3. The heart of the matter: a reduction to bilinear embedding estimate

To prove Theorem 1.2 we need a key inequality. It is an inequality established by Wittwer [40] (see also [26] on which]citeWit is based).

$$\sum_I |(\phi w, h_I)||(\psi\sigma, h_I)| \leq C[w]_{A_2}\,\|\phi\|_w \|\psi\|_\sigma\,. \tag{3.1}$$

In fact, if (3.1) is proved we just put

$$B_Q(X,Y,x,y,u,v) := \sup\{\frac{1}{|J|}\sum_{I\subseteq J} |(\phi w, h_I)||(\psi\sigma, h_I)| : \langle \phi^2 w\rangle_I = X, \langle \psi^2\sigma\rangle_I = Y,$$

$$\langle\phi\rangle_I = x, \langle\psi\rangle_I = y, \langle w\rangle_I = u, \langle\psi\rangle_I = v\}\,.$$

All properties (1.1)–(1.3) can be easily checked as soon as (3.1) is proved. We give here an easy proof of (3.1)–considerably easier than in [40].

Lemma 3.1. *Below I's are dyadic intervals. We have the following decomposition:*

$$h_I = \alpha_I h_I^w + \beta_I \frac{\chi_I}{\sqrt{I}}\,,$$

where
1) $|\alpha_I| \le \sqrt{\langle w\rangle_I}$,
2)$|\beta_I| \le \frac{|\Delta_I w|}{\langle \phi\rangle_I}$,
3) $\{h_I^w\}_I$ *is an orthonormal basis in* $L^2(w)$,
4) h_I^w *assumes on* I *two constant values, one on* I_+ *and another on* I_-.

We write

$$\sum_I |(\phi w, h_I)||(\psi\sigma, h_I)| \le$$

$$\sum_I |(\phi w, h_I^w)|\sqrt{\langle w\rangle_I}|(\psi\sigma, h_I^\sigma)|\sqrt{\langle\sigma\rangle_I} +$$

$$\sum_I |\langle\phi w\rangle_I \frac{|\Delta_I w|}{\langle w\rangle_I}|(\psi\sigma, h_I^\sigma)|\sqrt{\langle\sigma\rangle_I}\sqrt{I} +$$

$$\sum_I |\langle\psi\sigma\rangle_I \frac{|\Delta_I \sigma|}{\langle \sigma\rangle_I}|(\phi w, h_I^w)|\sqrt{\langle w\rangle_I}\sqrt{I} +$$

$$\sum_I |\langle\phi w\rangle_I|\langle\psi\sigma\rangle_I \frac{|\Delta_I w|}{\langle w\rangle_I}\frac{|\Delta_I \sigma|}{\langle \sigma\rangle_I}\sqrt{I}\sqrt{I} =: I + II + III + IV\,.$$

Obviously

$$I \le C[w]_{A_2}^{1/2}\|\phi\|_w\|\psi\|_\sigma\,. \tag{3.2}$$

Terms II, II are symmetric, so consider II. Using Bellman function one can prove now that

$$II \le C[w]_{A_2}\|\phi\|_w\|\psi\|_\sigma\,. \tag{3.3}$$

$$III \le C[w]_{A_2}\|\phi\|_w\|\psi\|_\sigma\,. \tag{3.4}$$

If we do the same in IV by using Cauchy's inequality, we would get

$$IV \le C[w]_{A_2}^{3/2}\|\phi\|_w\|\psi\|_\sigma\,,$$

which is not our coveted linear estimate. So I, II, II are fine and linear estimate of exterior sum $\overline{\sigma_{11e}}$ is equivalent to the linear estimate of IV.

4. Carleson measures built on $w \in A_2$ and their estimates

Let us introduce **bi-sublinear** sum

$$B(\phi w, \psi\sigma) := \sum_I |\langle \phi w\rangle_I| |\langle \psi\sigma\rangle_I| \frac{|\Delta_I w|}{\langle w\rangle_I} \frac{|\Delta_I \sigma|}{\langle \sigma\rangle_I} |I|\,.$$

Everything is reduced to the estimate of this **bi-sublinear** sum.

We can rewrite it as

$$\sum_I \frac{|\langle \phi w\rangle_I|}{\langle w\rangle_I} \frac{|\langle \psi\sigma\rangle_I|}{\langle \sigma\rangle_I} |\Delta_I w||\Delta_I \sigma||I| \le [w]_{A_2}\, \|\phi\|_{L^2(L,\sigma)} \|\psi\|_{L^2(L,\sigma)}\,. \tag{4.1}$$

This is immediately reductive to Carleson measure estimate. In fact, the LHS of (4.1) can be rewritten as

$$\sum_I \frac{|\langle \phi w\rangle_I|}{\langle w\rangle_I} \frac{|\langle \psi\sigma\rangle_I|}{\langle \sigma\rangle_I} |\Delta_I w||\Delta_I \sigma||I| \le B \int_L M_w\phi(x) M_\sigma\psi(x) dx\,, \tag{4.2}$$

where B is the Carleson norm of the measure given by the formula

$$\alpha_I = |\Delta_I w||\Delta_I(\sigma)||I|\,. \tag{4.3}$$

In fact, (4.2) is a simple geometric argument: exercise!

But the RHS of (4.2) is estimated by Cauchy inequality independently of $[w]_{A_2}$ (we learnt this other trick from [4]):

$$\int M_w\phi(x) M_\sigma\psi(x) dx = \int M_w\phi(x) M_\sigma\psi(x) \sqrt{w(x)}\sqrt{\sigma(x)} dx \le$$

$$\|M_w\phi\|_w \|M_\sigma\chi_L\|_\sigma \le A\, \|\phi\|_{L^2(w)} \|\psi\|_{L^2(\sigma)}\,.$$

Combining this with (4.2) we obtain that everything follows from

Theorem 4.1.

$$\|\{\alpha_I\}_I\|_{Carl} \le A\,[w]_{A_2}\,.$$

Proof. In the paper [38] it is shown that if for all $I \in D$ we have that two positive functions u, v satisfy

$$\langle u\rangle_I \langle v\rangle_I \le 1$$

then for any $L \in D$ we also have

$$\frac{1}{|L|} \sum_{I\in D, I\subset L} |\Delta_I u||\Delta_I v||I| \le A\, \sqrt{\langle u\rangle_L \langle v\rangle_L}\,.$$

Take our $w \in A_2$ and put $u = w/[w]_{A_2}, v = \sigma$. Then the assumption is satisfied, and we immediately get

$$\sum_{I\in D, I\subset L} |\Delta_I w||\Delta_I \sigma||I| \le A\,[w]_{A_2}^{1/2} \sqrt{\langle w\rangle_L \langle \sigma\rangle_L} |L|\,.$$

In particular, we obtain

$$\sum_{I\in D, I\subset L} |\Delta_I w||\Delta_I \sigma||I| \le A\,[w]_{A_2}\,|L|\,.$$

This is exactly (4.1) for measure $\{\alpha_I\}_I$! $\square$

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